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On a Shock Front in Burgers Turbulence

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We investigate how chaos propagates in the solution of Burgers equation $\partial_t u + u\,\partial_x u = 0$ with initial condition $u(\cdot, 0)$ distributed as a white noise on $\mathbb{R}^+$ and 0 on $\mathbb{R}^-$. We describe the evolution of the shock front that travels to the left. Asymptotics are given for both large and small time t.

KEY WORDS: Burgers turbulence; sticky particles; shock front.

1. INTRODUCTION

This work is related to the study of the solutions of Burgers/Riemann equation

$$\partial_t u + u\,\partial_x u = 0, \tag{1}$$

when the initial condition $u(\cdot, 0)$ is random. Even for very smooth initial conditions the solutions of (1) may develop shocks at finite time and become multi-stream. We select the "physical" solution by adding a viscosity term $\partial_t u + u\,\partial_x u = \epsilon\,\partial^2_{xx} u$ and then let ϵ tend to 0 ($\epsilon > 0$ corresponds to the inverse of Reynold's number). We obtain by this way the unique (weak) solution of (1) fulfilling some entropy conditions and usually called "entropy solution" of (1), see refs. 1–3. Burgers equation can be viewed as a simplified version of Navier–Stokes equation. If we think to the phenomenon of turbulence, it is interesting to understand the time-evolution of the solutions of (1) when the initial conditions are very chaotic. This is the starting point of a wide literature on solutions of (1) with random initial conditions. The case when $u(\cdot, 0)$ is distributed as a white noise has raised a special interest, see, for example, refs. 4–8. A white noise is the derivative

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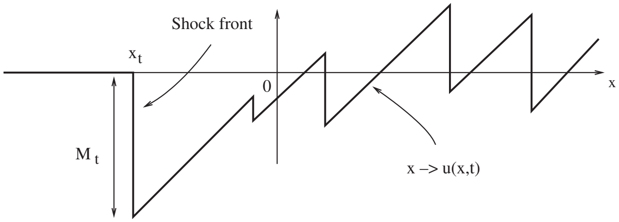

Fig. 1. Shape of $x \mapsto u(x, t)$.

(in a weak sense) of a Brownian motion. This is a very sharp initial condition, since it is not a real function but rather a distribution. It also corresponds to a completely uncorrelated initial data, which may be viewed has an analog in continuous space of random independent identically distributed (i.i.d.) variables. For those reasons, white noise appears as a "natural" model of chaos. We focus henceforth on the entropy solution of (1) with initial condition

$$u(\cdot, 0) = \begin{cases} \text{white noise} & \text{on} \quad]0, \infty[, \\ 0 & \text{on} \quad]-\infty, 0]. \end{cases} \tag{2}$$

The aim of this paper is to understand how chaos, initially at the right of 0, propagates to the left. The shape of the solution at a given time $t > 0$ is already well-known. There exists a strong shock, we shall call *shock front*, which propagates to the left. At its left $u(\cdot, t)$ equals 0. At its right $x \mapsto u(x, t)$ is a.s. a tooth-path, which is composed of pieces of line of slope $1/t$ separated by a discrete sequence of shocks, see Fig. 1.

Our goal is to describe the shock front, in particular its location x_t and its strength M_t. We must emphasize that the other characteristics of $x \mapsto u(x, t)$ are well-known. Indeed, if we write $x_n(t)$ and $M_n(t)/t$ for the location and the strength at time t of the nth shock at the right of the shock front, then $(x_n(t), M_n(t))_{n \in \mathbb{N}}$ is a Markov chain initialized by $(x_0(t), M_0(t)) = (x_t, M_t)$ with known transitions,[2] see Groeneboom[(9)] or Frachebourg and Martin.[(7)] The time evolution of the shocks at the right of x_t is the same as in the case where $u(\cdot, 0)$ is a white noise on the whole line $\mathbb{R}$. This evolution is depicted in refs. 8 and 10. Some other quantities have also been computed (for example the flux of mass crossing a given point $x \in \mathbb{R}$), we refer to refs. 7, 11, and 12 for recent works on the topic.

It is convenient to describe the solution of (1) with initial condition (2) in terms of a system of ballistic aggregation. This system involves the sticky

[2] In fact, Groeneboom[(9)] or Frachebourg–Martin[(7)] have computed these transitions for two-sided white noise initial velocity, but they remain unchanged in the one-sided case.

particles introduced by Zeldovich in astrophysics. Actually, Burgers equation (1) appears as a model for the study of the formation of the large scale structures of the universe, see refs. 13–15 and also ref. 16. Consider at time $t = 0$, particles of mass 1 on a one-dimensional lattice, say $\mathbb{Z}$. Assume that the particles at the left of 0 are at rest and give random i.i.d. velocities to the particles at the right of the origin. Then, let the system evolve according to the (deterministic) dynamic of free sticky particles: between collisions the particles move at constant speed, and when some of them meet, they stick and merge into a new particle whose mass, respectively momentum, equals the sum of the masses, respectively momenta, of the particles which have collide. Such collisions dissipate energy. It is remarkable that as soon as the law of the initial velocities of the particles initially in $\mathbb{Z}^+$ is centered with finite variance, the hydrodynamic limit of this system can be completely described in terms of the solution of (1) with initial condition (2), see ref. 17 for proof. Indeed, the velocity field at time t of the hydrodynamic limit of the previous system exactly corresponds to the solution $u(\cdot, t)$ at time t of (1) with initial velocity (2). Moreover, the macroscopic clusters (=clusters of positive mass) of the ballistic model are isomorphic to the shocks of $x \mapsto u(x, t)$. More precisely, the locations of the macroscopic clusters exactly correspond to the locations of the shocks of $x \mapsto u(x, t)$, and the masses of the clusters correspond to the strength of the shocks. The shape of the system at time $t > 0$ is thus the following. There a massive *front cluster* traveling to the left. Its location is x_t and its mass M_t. At its left there are infinitesimal particles at rest, which do not have yet been perturbed by the turbulence. At its right all particles have clump into macroscopic clusters, whose locations $(x_n(t))_{n \in \mathbb{N}}$ form a discrete sequence of $[x_t, \infty[$, see Fig. 2. Their masses are given by $(M_n(t))_{n \in \mathbb{N}}$. We shall mainly use thenceforth this description of the solution of (1) with initial condition (2), since it is more natural to speak of collisions or aggregation of clusters than of collisions or aggregation of shocks.

The rest of the paper is organized as follows. In Section 2, we recall some standard features on Burgers turbulence. Section 3 is devoted to the calculation of the statistics of the shock front at a fixed time t. In the last section, the temporal evolution of the shock front is depicted and asymptotics are computed for large or small time t.

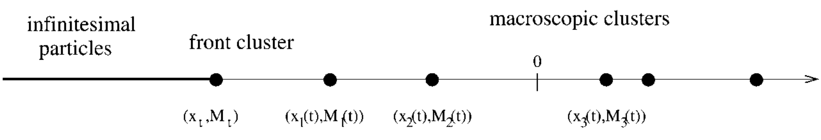

Fig. 2. Shape of the system at time $t > 0$.

2. PRELIMINARIES

2.1. Standard Features on Burgers Turbulence

As soon as the initial potential W satisfies the condition $W_z \ll z^2$ when $|z| \to \infty$ (which is satisfied with probability one when W is a Brownian motion), it is well-known that the state of the system can be fully described in terms of W. More precisely, every physical quantities may be expressed in terms of the right-most location $a(x, t)$ of the minimum of $z \mapsto W_z + \frac{1}{2t}(z-x)^2$ on $\mathbb{R}$. In a geometrical point of view, $a(x, t)$ may be interpreted as follows. Bring up a parabola $z \mapsto -\frac{1}{2t}(z-x)^2 + C$ until it touches the graph of the initial potential $z \mapsto W_z$. Then $a(x, t)$ corresponds to the abscissa of the right-most contact point, see Fig. 3. Physically, $a(x, t)$ represents the right-most initial location of the particles that lie in $]-\infty, x]$ at time t. A macroscopic cluster is then located at $x \in \mathbb{R}$, if and only if $a(x, t) > a(x-, t) := \lim_{z \uparrow x} a(z, t)$, or in other words if and only if the parabola $z \mapsto -\frac{1}{2t}(z-x)^2 + C$ touches the initial potential $z \mapsto W_z$ at at least two points. The mass of this cluster then equals $a(x, t) - a(x-, t)$, and its velocity is enforced by the conservation of momentum, viz

$$u(x, t) = \frac{1}{a(x, t) - a(x-, t)} (W_{a(x, t)} - W_{a(x-, t)}) = \frac{1}{2t}(2x - a(x, t) - a(x-, t)),$$

see Fig. 3 for the geometrical interpretation of these quantities. Finally, the velocity field is given at any point $x \in \mathbb{R}$ where $z \mapsto a(z, t)$ is continuous, by the celebrated Hopf–Cole formula (see refs. 2 and 3)

$$u(x, t) = \frac{1}{t}(x - a(x, t)).$$

The initial condition (2) corresponds to an initial potential

$$W = \begin{cases} \text{Brownian motion} & \text{on} \quad]0, \infty[, \\ 0 & \text{on} \quad]-\infty, 0]. \end{cases}$$

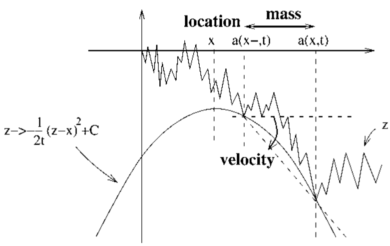

Fig. 3. Geometrical interpretation of the physical quantities.

It is known that there exists in this case a shock front, at the right of which the function $x \mapsto a(x, t)$ is a.s. a step function and one says that the shock structure is discrete. This exactly corresponds to the fact that all particles at the right of x_t have merge into a discrete sequence of macroscopic clusters. At the left of the shock front $a(x, t) = x$, which means that in $]-\infty, x_t[$ the density of mass equals 1 and the velocity field 0. We thus have at the left of x_t infinitesimal particles at rest, uniformly spread on $]-\infty, x_t[$.

2.2. Analyzing the Shock Front

We focus henceforth on the shock front and call x_t its location. The front cluster consists of the particles initially in $[x_t, a(x_t, t)]$. The mass M_t of the front cluster thus equals $m_t - x_t$, with the notation $m_t = a(x_t, t)$. These quantities may be interpreted in geometrical terms of the initial potential W as follows. Take a parabola $z \mapsto -\frac{1}{2t}(z-x)^2$ and let x increase from $-\infty$ until the parabola touches the graph of the Brownian motion $(W_z; z \geqslant 0)$. Then x_t is the value of x for which the parabola touches the initial potential, and m_t is the abscissa of the contact point, see Fig. 3. The ordinate p_t of the contact point represents the momentum of the front cluster. A glance at Fig. 4 shows that

$$p_t = -\frac{1}{2t}(m_t - x_t)^2 = -\frac{1}{2t} M_t^2.$$

It is interesting for our analysis to interpret these miscellaneous quantities in terms of the first passage time $T_y := \inf\{z \geqslant 0; W_z \leqslant -y\}$ of the Brownian motion below the level $-y$. We recall that T is a subordinator, viz an increasing Markov process with stationary and independent

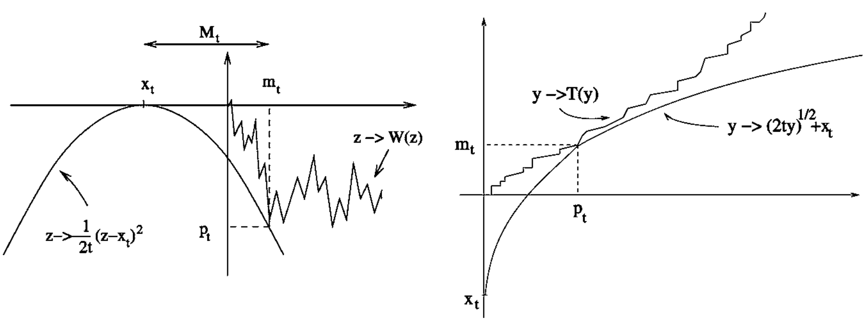

Fig. 4. Geometrical interpretations of the shock front.

increments. A moment of thought shows that m_t takes value in the set $\mathscr{M} := \{z \geqslant 0; W_z = \inf_{0 \leqslant u \leqslant z} W_u\}$. The process $y \mapsto T_y$ is the right-continuous inverse of $z \mapsto \inf_{0 \leqslant u \leqslant z} W_u$, so that x_t and p_t corresponds to the minimum and the right-most location of the minimum of $y \mapsto T_y - \sqrt{2ty}$, see Fig. 4. This minimum is achieved since, according to the law of the iterated logarithm,

$$\liminf_{y \to \infty} \frac{2 \log \log y}{y^2} T_y = 1 \quad \text{a.s.},$$

see, for example, III.5 Theorem 14 in ref. 18.

3. STATISTICS AT A FIXED TIME *t*

The scaling property $(T_{\lambda y}; y \geqslant 0) \overset{\text{law}}{=} (\lambda^2 T_y; y \geqslant 0)$ of the process $y \mapsto T_y$ entails the equality in law

$$(t^{2/3}(T_{t^{-1/3}y} - \sqrt{2t^{-1/3}y}); y \geqslant 0) \overset{\text{law}}{=} (T_y - \sqrt{2ty}; y \geqslant 0).$$

According to the above geometrical analysis of the system, the parameters of the shock front follow the scaling property

$$(x_t, p_t, M_t, m_t) \overset{\text{law}}{=} (t^{2/3} x_1, t^{1/3} p_1, t^{2/3} M_1, t^{2/3} m_1). \tag{3}$$

We can thus focus on time $t = 1$. The shock front is then completely described by the variables (x_1, M_1), since $p_1 = -\frac{1}{2} M_1^2$ and $m_1 = M_1 + x_1$. The law of (x_1, M_1) can be easily computed from the work of Groeneboom.[9]

We denote by $\mathtt{Ai}$ and $\mathtt{Bi}$ the Airy functions (see ref. 19 on p. 446 for the definition) and following Groeneboom we introduce the functions $\Lambda: \mathbb{R} \to \mathbb{R}^+$ and $g: \mathbb{R} \to \mathbb{R}^+$ which have Fourier transform

$$\hat{\Lambda}(\lambda) = \int_{\mathbb{R}} e^{i\lambda s} \Lambda(s)\, ds = \pi 2^{1/3} \left(\mathtt{Bi}(i2^{1/3}\lambda + 2^{-2/3}x^2) - \frac{\mathtt{Bi}(i2^{1/3}\lambda)\, \mathtt{Ai}(i2^{1/3}\lambda + 2^{-2/3}x^2)}{\mathtt{Ai}(i2^{1/3}\lambda)} \right) \quad \text{and}$$

$$\hat{g}(\lambda) = \int_{\mathbb{R}} e^{i\lambda s} g(s)\, ds = \frac{2^{1/3}}{\mathtt{Ai}(i2^{-1/3}\lambda)}.$$

We also write $h(m,\cdot)\colon \mathbb{R}^+ \to \mathbb{R}^+$ for the function either defined by its Laplace transform

$$\int_0^\infty e^{-\lambda x} h(m, x)\, dx = \frac{\mathtt{Ai}(2^{2/3}m + 2^{-1/3}\lambda)}{\mathtt{Ai}(2^{-1/3}\lambda)},$$

or by the series

$$h(m, x) = 2^{1/3} \sum_{n=1}^{\infty} \frac{\mathtt{Ai}(2^{2/3}m - \omega_n)}{\mathtt{Ai}'(-\omega_n)} \exp(-2^{1/3} x \omega_n),$$

where $0 > -\omega_1 > -\omega_2 > \cdots$ denotes the zeros of $\mathtt{Ai}$ ranked in decreasing order.

Theorem 1 (Statistics at Time $t = 1$). In the above notation, the law of x_1 and (x_1, M_1) are given by the following formulaes

$$\mathbb{P}(x_1 > -x) = e^{-x^3/3} \Lambda(x) \qquad \text{and} \tag{4}$$

$$\mathbb{P}(-x_1 \in dx, M_1 \in dM) = \frac{M}{2}\, e^{-x^3/3} g(2^{-2/3}M)\, h(2^{-4/3}x^2, 2^{-2/3}(M - x)), \tag{5}$$

for $M, x > 0$.

Proof of Theorem 1. The event $\{x_1 > -x\}$ exactly corresponds to the event $\{W_z \geqslant -\frac{1}{2}(z+x)^2;\ \forall z \geqslant 0\}$, and the probability of the latter is given by Theorem 3.1 in ref. 9. We focus now on the joint law of (x_1, M_1). We write $\mathscr{M}_x = \min\{W_z + \frac{1}{2}(z+x)^2;\ z \geqslant 0\}$ and τ_x for the largest abscissa where this minimum is achieved. We can express the probability density of (x_1, M_1) in terms of $\mathscr{M}_x$ and τ_x, indeed

$$\mathbb{P}(-x_1 \in dx, M_1 \in dM) = \lim_{h \downarrow 0} \frac{1}{h}\, \mathbb{P}(\mathscr{M}_{x+h} > 0, \mathscr{M}_x < 0, \tau_x + x \in dM)\, dx. \tag{6}$$

Notice that $\tau_{x+h} \leqslant \tau_x$, so that

$$\mathscr{M}_{x+h} = \min\left\{W_z + \tfrac{1}{2}(z+x)^2 + h(z+x) + \tfrac{1}{2}h^2;\ z \geqslant 0\right\}$$

$$= \begin{cases} = \mathscr{M}_x + h(\tau_x + x) + h^2/2, & \text{when} \quad \tau_{x+h} = \tau_x, \\ \leqslant \mathscr{M}_x + h(\tau_x + x) + h^2/2, & \text{else.} \end{cases}$$

It can be shown that the probability density $\mathbb{P}(\mathcal{M}_{x+h} < 0, \mathcal{M}_x > 0, \tau_{x+h} + x < M, \tau_x + x \in dM)/\, dM$ is negligible with respect to h, for example with the following inequalities

$$\begin{aligned}
&\mathbb{P}(\mathcal{M}_{x+h} < 0, \mathcal{M}_x > 0, \tau_{x+h} + x < M, \tau_x + x \in dM)/dM \\
&\quad \leqslant \mathbb{P}\left(-\frac{h^2}{2} - hM < \mathcal{M}_x < 0, \tau_{x+h} + x < M, \tau_x + x \in dM\right)\Big/ dM \\
&\quad \leqslant \underbrace{\mathbb{P}\left(-\frac{h^2}{2} - hM < \mathcal{M}_x < 0, \tau_x + x \in dM\right)\Big/ dM}_{\overset{h\to 0}{=} O(h)} \\
&\qquad \times \underbrace{\mathbb{P}\left(\tau_{x+h} + x < M \,\middle|\, -\frac{h^2}{2} - hM < \mathcal{M}_x < 0, \tau_x + x = M\right)}_{\overset{h\to 0}{=} o(1)}.
\end{aligned}$$

The calculation of the limit (6) is now straightforward:

$$\begin{aligned}
&\mathbb{P}(\mathcal{M}_{x+h} > 0, \mathcal{M}_x < 0, \tau_x + x \in dM)/dM \\
&\quad \sim \mathbb{P}\left(-\frac{h^2}{2} - hM < \mathcal{M}_x < 0, \tau_x + x \in dM\right)\Big/ dM \\
&\quad \sim h\, M \mathbb{P}(\mathcal{M}_x \in d0, \tau_x + x \in dM)/d0\, dM,
\end{aligned}$$

when $h \to 0+$. The probability density of $\mathcal{M}_x$ and τ_x is given by Corollary 3.1 in ref. 9 and putting pieces together one obtains formula (5). ▮

4. TIME EVOLUTION OF THE SHOCK FRONT

The dynamic of the shock front (or front cluster) is governed by two phenomena. Its movement to the left is continuously braked by the infinitesimal particles at rest on its left, but sometimes a macroscopic cluster on its right, catches it and increases sharply its velocity.

It follows from the identity $p_t = -\frac{1}{2t} M_t^2$ that the time evolution of the cluster can be completely described in terms of $t \mapsto M_t$. For example, the time evolution of x_t is given by

$$x_t = -\frac{1}{2} \int_0^t M_s \frac{ds}{s}.$$

Next theorem characterizes the process $t \mapsto M_t$. It involves again the function g of the previous section but also the function p: $\mathbb{R}^+ \to \mathbb{R}^+$ defined by

$$p(x) = 2 \sum_{k=1}^{\infty} \exp(-2^{1/3}\omega_k x),$$

where, as before, $0 > -\omega_1 > -\omega_2 > \cdots$ represents the zeros of `Ai` ranked in decreasing order.

Theorem 2 (Statistics of the Dynamics). The mass $t \mapsto M_t$ of the front cluster is an increasing (non homogeneous) Markov process with infinitesimal generator at time $t > 0$

$$G_t f(M) = \frac{M}{2t} f'(M) + \int_0^{\infty} (f(M+m) - f(M)) \times \frac{m(M+m)}{4t^3} p((2t)^{-2/3} m) \frac{g((2t)^{-2/3}(M+m))}{g((2t)^{-2/3} M)} \, dm,$$

for $M > 0$ and $f: \mathbb{R}^+ \to \mathbb{R}$ with derivatives of any order and compact support.

Proof of Theorem 2. The momentum $|p_t|$ of the front cluster corresponds to the right-most location of the minimum of $y \mapsto T_y - \sqrt{2ty}$, see Section 2. It is then standard that $t \mapsto p_t$ is a Markov process. Indeed, a path decomposition due to Millar[20] entails that conditionally on $|p_t| = p$, the processes $(T_y, y \leqslant |p_t|)$ and $(T_{|p_t|+y} - T_{|p_t|}, y \geqslant 0)$ are independent. For $t_1 < t < t_2$ $|p_{t_1}|$ can be viewed as the location of the minimum of $(T_y - \sqrt{2t_1 y}; y \leqslant |p_t|)$ and $|p_{t_2}| - |p_t|$ as the location of the minimum of $(T_{|p_t|+y} - T_{|p_t|} - \sqrt{2t_2(y + |p_t|)}; y \geqslant 0)$. The Markov property of p_t and also of $M_t = \sqrt{2t\,|p_t|}$ follows.

Assume that between time t and time $t+s$ the path $u \to M_u$ is continuous, which means that during this time interval no macroscopic clusters aggregate to the front cluster. According to the law of conservation of momentum $p_t = p_{t+s}$, the mass of the front cluster at time $t+s$ is

$$M_{t+s} = \sqrt{1 + \frac{s}{t}}\, M_t.$$

Let us estimate now the contribution in the growth of M_t of the collisions of macroscopics clusters coming from the right. We need to compute the probability that a cluster of mass $\approx m$ sticks to the front cluster at time $\approx t$.

Lemma 1. When $h, \eta \to 0+$, we have the asymptotic

$$\mathbb{P}(m \leqslant M_{t+h} - M_t \leqslant m+\eta \mid M_t = M) \sim h\,\eta\,\frac{m(m+M)}{4t^3}\,p((2t)^{-2/3}m)\,\frac{g((2t)^{-2/3}(M+m))}{g((2t)^{-2/3}M)}.$$

The asymptotic of Lemma 1 is obtained in connecting $\mathbb{P}(m \leqslant M_{t+h} - M \leqslant m+\eta \mid M_t = M)$ to the rate of jump of $x \mapsto a(x,t)$, which have been computed by Groeneboom (see ref. 9, Theorem 4.1). Assume that when $h, \eta \to 0+$

$$\mathbb{P}(m \leqslant M_{t+h} - M \leqslant m+\eta \mid M_t = M) \sim \mathbb{P}(a(h(M+m)/2t, t) - a(0,t) \in [m, m+\eta] \mid a(0,t) = M). \quad (7)$$

As in ref. 11, p. 558, we have

$$\mathbb{P}(a(h(M+m)/2t, t) - a(0,t) \in [m, m+\eta] \mid a(0,t) = M) \overset{h,\eta\to 0+}{\sim} h\,\eta\,\frac{m(m+M)}{4t^3}\,p((2t)^{-2/3}m)\,\frac{g((2t)^{-2/3}(M+m))}{g((2t)^{-2/3}M)},$$

and Lemma 1 follows. A rigorous proof of formula (7) can be obtained by adapting the proof of Lemma 1 in ref. 11 to our case. We omit the details and just sketch the argument. The shock structure at the right of the front shock is discrete, so that we can forget the effects of the collisions during small time intervals. A cluster of mass $\approx m$ and velocity $\approx v$ collides with the front cluster during the time interval $[t, t+h]$, "if and only if" the first cluster at the right of x_t has mass $\approx m$, velocity $\approx v$ and is located in $[x_t, x_t - (v - v_t)\,h]$, where $v_t < 0$ denotes the velocity of the front cluster. The first cluster at the right of x_t is build from the particles initially in the interval $]m_t, m_t + m[$. Up to time t, these particles do not have interacted with the other particles so that, according to the conservation of momentum, the velocity of their center of mass is constant et equals therefore v. At the initial time the center of mass is located at $\frac{1}{2}(2m_t + m)$, whereas at time t it is located at $\approx x_t$. Velocity v thus equals

$$v \approx \frac{1}{t}\left(x_t - \frac{1}{2}(2m_t + m)\right) \approx -\frac{1}{2t}(2M_t + m).$$

Since $v_t = -\frac{1}{2t}M_t$ the event "a cluster of mass $\approx m$ collide with the front cluster during the time interval $[t, t+h]$" corresponds roughly to the event $a(x_t + \frac{h}{2t}(M_t + m), t) - a(x_t, t) \approx m$. According to the Markov property of

$x \to a(x, t)$, this event is distributed conditionally on $M_t = M$ as the event $a(\frac{h}{2t}(M+m), t) - a(0, t) \approx m$ conditionally on $a(0, t) = M$. ▮

We can evaluate the infinitesimal generator G_t at time $t > 0$ of $t \mapsto M_t$. For f with compact support and derivatives of any order and $M > 0$, $G_t f(M)$ is defined as

$$\lim_{h \to 0+} \frac{1}{h} (\mathbb{E}(f(M_{t+h}) \mid M_t = M) - f(M))$$

$$= \lim_{h \to 0+} \frac{1}{h} \mathbb{E}(f(M_{t+h}) - f(M_t) \mid M_t = M).$$

Recall that when $t \to M_t$ does not jump during the time interval $[t, t+h]$, then $M_{t+h} = \sqrt{1 + \frac{h}{t}}\, M_t$, so that

$$\frac{1}{h} \mathbb{E}(f(M_{t+h}) - f(M_t) \mid M_t = M)$$

$$\overset{h \to 0+}{\sim} \frac{1}{h}\left(f\left(\sqrt{1+\frac{h}{t}}\, M\right) - f(M)\right)$$

$$+ \frac{1}{h} \int_{M(\sqrt{1+h/t}-1)}^{\infty} (f(M+m) - f(M))\, \mathbb{P}(M_{t+h} - M_t \in dm \mid M_t = M)$$

$$\overset{h \to 0+}{\sim} \frac{M}{2t} f'(M) + \int_0^{\infty} (f(M+m) - f(M))$$

$$\times \frac{m(m+M)}{4t^3}\, p((2t)^{-2/3} m)\, \frac{g((2t)^{-2/3}(M+m))}{g((2t)^{-2/3} M)}\, dm.$$

The proof of Theorem 2 is complete. ▮

Remark. Call T^* the first time after time t at which a macroscopic cluster collides with the front cluster and m^* the mass of this cluster. One may be interested in evaluating the law of (T^*, m^*). It is an easy task when one follows the argument of the proof of Theorem 1 in ref. 11. Using the Markov property of $t \mapsto M_t$, the conditional probability density of (T^*, m^*) given $M_t = M$ can be written as

$$\mathbb{P}(T^* \in ds, m^* \in dm \mid M_t = M)$$

$$= \mathbb{P}(T^* \geqslant s \mid M_t = M)\, \mathbb{P}\left(T^* \in ds, m^* \in dm \mid M_s = \sqrt{\frac{s}{t}}\, M\right).$$

As in ref. 11, the first term equals

$$\mathbb{P}(T^* \geqslant s \mid M_t = M) = \lim_{y \to 0+} \frac{\frac{\partial}{\partial \alpha} \mathbb{P}^\alpha \left(W_z \geqslant -\frac{1}{2s} z \left(z + 2M \sqrt{\frac{s}{t}} \right); \forall z \geqslant 0 \right)}{\frac{\partial}{\partial \alpha} \mathbb{P}^\alpha \left(W_z \geqslant -\frac{1}{2t} z(z+2M); \forall z \geqslant 0 \right)}$$

$$= \left(\frac{t}{s}\right)^{1/3} \exp\left(-\frac{M^3}{6} \left(\frac{1}{t^2} - \frac{1}{\sqrt{t^3 s}} \right) \right) \frac{g(2^{-2/3} s^{-1/6} t^{-1/2} M)}{g((2t)^{-2/3} M)}.$$

The second term can be evaluate thanks to Lemma 1:

$$\mathbb{P}\left(T^* \in ds, m^* \in dm \mid M_s = \sqrt{\frac{s}{t}}\, M \right)$$

$$= dm\, ds \times \frac{m \left(\sqrt{\frac{s}{t}}\, M + m \right)}{4 s^3}\, p((2s)^{-2/3} m)\, \frac{g\left((2s)^{-2/3} \left(\sqrt{\frac{s}{t}}\, M + m \right) \right)}{g(2^{-2/3} s^{-1/6} t^{-1/2} M)}.$$

Combining the two previous expressions yields the formula

$$\mathbb{P}(T^* \in ds, m^* \in dm \mid M_t = M)$$

$$= \frac{t^{1/3}\, m \left(\sqrt{\frac{s}{t}}\, M + m \right)}{4\, s^{10/3}} \exp\left(-\frac{M^3}{6} \left(\frac{1}{t^2} - \frac{1}{\sqrt{t^3 s}} \right) \right)$$

$$\times p((2s)^{-2/3} m)\, \frac{g\left((2s)^{-2/3} \left(\sqrt{\frac{s}{t}}\, M + m \right) \right)}{g((2t)^{-2/3} M)}\, ds\, dm.$$

One may also be interested in having the asymptotic behaviour of x_t when t tends to 0 or ∞.

Proposition 4.1 (Asymptotics of x_t). At large time t we have, with probability one, the asymptotics:

$$\limsup_{t \to \infty} \frac{-x_t}{(t^2 \log \log t)^{1/3}} = \frac{2^{5/3}}{3}, \qquad \text{and} \tag{8}$$

$$\lim_{t \to \infty} \frac{-x_t}{t^{2/3}} (\log t)^{2/3+\delta} = \infty, \qquad \forall\, \delta > 0. \tag{9}$$

Similar asymptotics hold for small time t. Indeed, the same argument shows that a.s.

$$\limsup_{t\to 0} \frac{-x_t}{(t^2 \log|\log t|)^{1/3}} = \frac{2^{5/3}}{3}, \qquad \text{and}$$

$$\lim_{t\to 0} \frac{-x_t}{t^{2/3}} |\log t|^{2/3+\delta} = \infty, \qquad \forall\, \delta > 0.$$

Proof of Proposition 1. The asymptotics of x_t derive from the law of the iterated logarithm for the process $y \to T_y$

$$\liminf_{y\to\infty} T_y \frac{\log\log y}{y^2} = \frac{1}{2} \quad \text{a.s.,} \tag{10}$$

and for any $\delta > 0$

$$\lim_{y\to\infty} \frac{T_y}{y^2(\log y)^{2+\delta}} = 0 \quad \text{a.s.,} \tag{11}$$

see, for example, III.5 Theorem 14 in ref. 18. We first focus on formula (8). Write for $t > 0$

$$\psi(t) = \frac{3}{2^{5/3}}\, t\, (\log\log t)^{1/3},$$

and for any $\epsilon > 0$,

$$\psi_+^*(y) = \sup_{t\geqslant 0} \{-(1+\epsilon)\,\psi(t) + \sqrt{2\,ty}\}.$$

Call t_0 the location of this supremum. It is the solution of the equation

$$-(1+\epsilon)\frac{2}{3}\, a\, t_0^{-1/3}(\log\log t_0)^{1/3}\left(1+\frac{1}{2\log t_0 \log\log t_0}\right) + \sqrt{\frac{y}{2t_0}} = 0,$$

where $a = \frac{3}{2^{5/3}}$. When y tends to infinity, then so does t_0, and conversely. Moreover, we then have the estimate

$$\sqrt{\frac{y}{2}} \sim \frac{2}{3}(1+\epsilon)\, a t_0^{2/3} (\log\log t_0)^{1/3}. \tag{12}$$

In particular, for large t_0 we can estimate $\psi(t_0)$ in terms of y and then obtain

$$\psi_+^*(y) \overset{y\to\infty}{\gtrsim} \frac{1}{(1+\epsilon)^3} \times \frac{y^2}{2\log\log y}.$$

Combining this estimate with (10) yields to the formula

$$\liminf_{y\to\infty} \frac{T_y}{\psi_+^*(y)} = (1+\epsilon)^3, \quad \text{a.s.}$$

As a consequence, with probability one and for y large enough $\psi_+^*(y) < T_y$, which implies that for t large enough $x_t > -(1+\epsilon)\,\psi(t)$ and thus that

$$\limsup_{t\to\infty} \frac{-x_t}{\psi(t)} \leqslant 1+\epsilon, \quad \text{a.s.} \tag{13}$$

Substitute now $-\epsilon$ to ϵ. The function

$$\psi_-^*(y) = \sup_{t\geqslant 0} \{-(1-\epsilon)\,\psi(t) + \sqrt{2\,ty}\}.$$

satisfies the equality

$$\liminf_{y\to\infty} \frac{T_y}{\psi_-^*(y)} = (1-\epsilon)^3, \qquad \text{a.s.},$$

so that there exists a.s. an increasing sequence $y_n \to \infty$ fulfilling the inequality $\psi_-^*(y_n) > T_{y_n}$. Call t_n the location of the supremum of $t \mapsto -(1-\epsilon)\,\psi(t) + \sqrt{2\,ty_n}$. The estimate (12) is transformed in

$$\sqrt{\frac{y_n}{2}} \overset{n\to\infty}{\gtrsim} \frac{2}{3}(1-\epsilon)\, a\, t_n^{2/3} (\log\log t_n)^{1/3},$$

which forces t_n to tends to infinity when $n \to \infty$. Notice moreover that $x_{t_n} < -(1-\epsilon)\,\psi(t_n)$, so that

$$\limsup_{t\to\infty} \frac{-x_t}{\psi(t)} \geqslant 1-\epsilon, \quad \text{a.s.}, \tag{14}$$

and let ϵ tend to 0 in bound (13) and (14) to obtain

$$\limsup_{t\to\infty} \frac{-x_t}{\psi(t)} = 1, \quad \text{a.s.}$$

Formula (9) can be obtained following the same way. Write for $t, \lambda > 0$

$$\phi(t) = \frac{t^{2/3}}{(\log t)^{2/3+\epsilon}} \qquad \text{and}$$

$$\phi_\lambda^*(y) = \sup_{t \geqslant 0} \{-\lambda\, \phi(t) + \sqrt{2ty}\}.$$

The asymptotics of $\phi_\lambda^*(y)$ for large y are given by

$$\phi_\lambda^*(y) \overset{y \to \infty}{\gtrsim} \frac{3^{5+3\epsilon}}{\lambda^3\, 2^6}\, y^2\, (\log y)^{2+3\epsilon}.$$

A glance at formula (11) then shows that

$$\lim_{y \to \infty} \frac{T_y}{\phi_\lambda^*(y)} = 0, \quad \text{a.s.},$$

which implies that $x_t < -\lambda\, \phi(t)$ for t large enough. In other words,

$$\liminf_{t \to \infty} \frac{-x_t}{t^{2/3}} (\log t)^{2/3+\epsilon} \geqslant \lambda, \qquad \text{a.s. for any} \quad \lambda > 0.$$

Let λ tend to infinity to obtain formula (9). ▮

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